УДК 621.865.8

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GEOMETRIC, KINEMATIC AND DYNAMIC MODELING OF CARTESIAN ROBOT

The goal of modeling any robot is to achieve the function for which it was designed in perfectly manner. When we model a robot we aim to find the equations that govern the robot geometric, kinematic and dynamic variables. In this paper, we studied the Cartesian robot (three prismatic D.O.Fs) to find the geometric and kinematic and dynamic models.

First, we use the geometric model to find workspace. Then, we find linear speeds of robot's links by using the kinematic model, we also study singular configurations. Finally, a dynamic model is carried out to calculate the forces and torques that are used to choose the motors for robot joints.

Key words: geometric modeling, kinematic modeling, dynamic modeling, cartesian robot.

Due to the growing importance of automated robot manipulators, colleges and institutes raced to teach and develop Robotic science. This science studies the design and modeling and control of mechanical systems. Figure 1 shows the distribution of the global robotics industry ratios, where we find that Japan has the largest ratio in the global robotics industry, followed by Europe and the United States. In the rest of the world we found that the robots industry does not have the required attention [1].

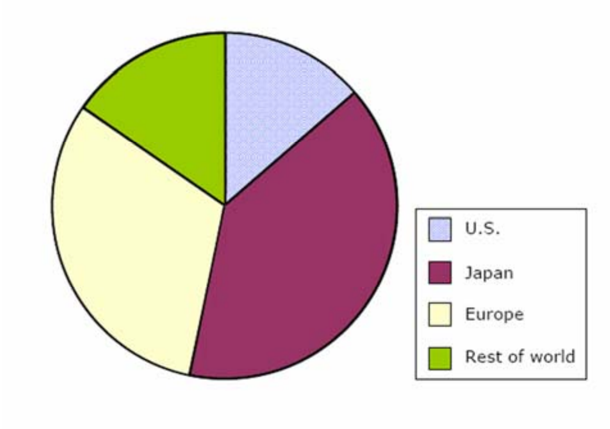

Figure 1. Percentage distribution of the world's robot industry

Despite of the importance of the robots, very few papers about robotic were published in Syria. This paper was written to use the theoretical knowledge of Robotic science to accomplish the Cartesian robot.

This paper includes the geometric modeling of the Cartesian robot with the characterization of the workspace. Then, a kinematic model was carried out in which velocities are calculated for robot's links, with the discussion of singular configurations. The paper also contains a dynamic modeling of the robot that provides the forces and torques that must be delivered by the motors that move the robot.

Robot characterization and its kinematic scheme

Figure 2 shows drawing of a three-dimensional Cartesian robot (designed using Inventor). A robot consists of links that symbolize by C_i (Corp) and joints that symbolize by L_i (Liaison) and $i = \{0, 1, 2, \text{and } 3\}$. The first Prismatic joint between the two links C_0 and C_1 allows C_1 to move up and down, the second one is the joint between two links C_1 and C_2 which allows C_2 to move right and left. The last joint between two links C_2 and C_3 allows C_3 to move forward and backward. The numerical values of link's mass are:

$$M_1 = 10 \text{ kg}, \quad M_2 = 4.4 \text{ kg}, \quad M_3 = 5 \text{ kg}, \quad M_L = 30 \text{ kg}.$$

The numerical values of friction coefficients in robot's joints are:

$$\mu_1 = \mu_2 = \mu_3 = 0.25,$$

where M_i – i^{th} link mass; M_L – load mass; μ_i – friction coefficient in the i^{th} joint.

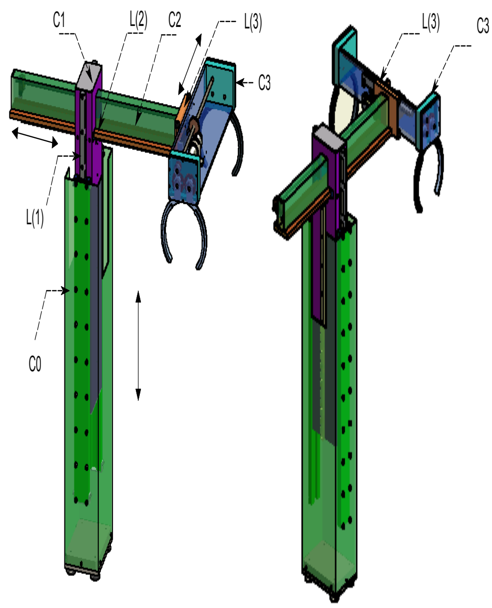

Figure 2. A three dimensional drawing of the Cartesian robot (by Inventor)

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Kinematic chain of the robot is shown in Figure 3.

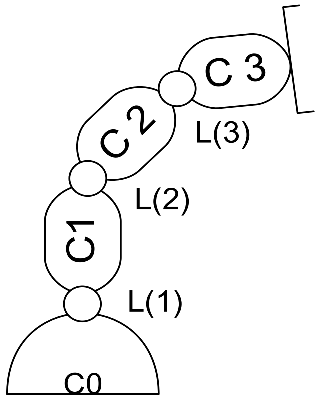

Figure 3. Kinematic chain of the robot

Specifications of the previous chain:

- number of links: $L = 4$;
- number of joints: $J = 3$.

So we can find the mobility M by Gruebler's equation [2]:

$$M = 3(L-1) - 2J. \qquad (1)$$

$M = 3$ so we need three motors to move the robot, one motor at each one of the joints. Therefore vector of motorized joint coordinates is: $\mathbf{q} = \begin{pmatrix} r_1 & r_2 & r_3 \end{pmatrix}^T$.

Figure 4 shows the kinematic scheme of the robot and the links which it is composed of.

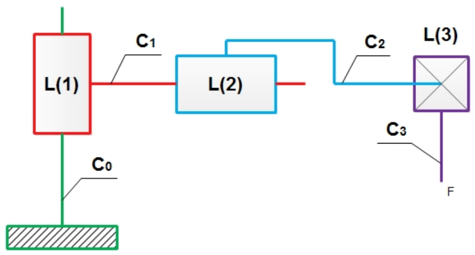

Figure 4. The kinematic scheme of the robot

Where C_i is the i^{th} link; L_i is the i^{th} joint; F is Terminal link.

Geometric modeling

The aim of this modeling is giving location of the terminal link in terms of the joint coordinates of the robot, and identifying workspace.

Note in Figure 5 that links are connected with each other by the following joints:

$L(1)$: simple prismatic joint between C_0 and C_1 along $\mathbf{Z_1}$.

$L(2)$: simple prismatic joint between C_1 and C_2 along $\mathbf{Z_2}$.

$L(3)$: simple prismatic joint between C_2 and C_3 along $\mathbf{Z_3}$.

Depending on Denavit and Hartenberg, method, which will be explained in the following section, we choose the positioning of the coordinates frames associated with Robot's links as shown in Figure 5.

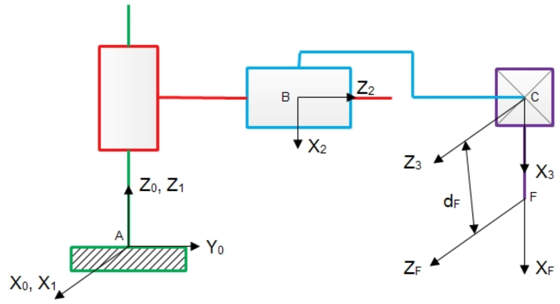

Figure 5. The coordinates frame associated with each of the robot's link

Denavait-Hartenberg method

Denavit and Hartenberg placed a modeling method in 1955 for the robot manipulators (open chain) based on transformation in homogeneous coordinates. The aim of this method is to find a way to unify the frames placed on the robot's links [3, 4, 5]. According to this method, we number links from 0 to n and put frames as follows:

- choose axis $\mathbf{Z_i}$ is along the axis of the joint i;
- choose axis $\mathbf{X_i}$ is along the common perpendicular with axes $\mathbf{Z_i}$ and $\mathbf{Z_{i+1}}$.

Also fix frame R_F with terminal link.

We find geometric parameters for the robot shown in Table 1.

Table 1. **Geometric parameters for the robot**

i	σ_i	α_i	d_i	θ_i	r_i
1	1	0	0	0	r_1
2	1	–90°	0	90°	r_2
3	1	90°	0	0	r_3
F	–	0	d_F	0	0

Where we have the following variables:

- σ_i : indicate the type of joint so that
- $\sigma_i = 0$ when the joint is revolute;
- $\sigma_i = 1$ when the joint is prismatic;
- α_i : angle between axes $\mathbf{Z_{i-1}}$, $\mathbf{Z_i}$ corresponding to rotation about $\mathbf{X_{i-1}}$;
- d_i : distance between $\mathbf{Z_{i-1}}$, $\mathbf{Z_i}$ along $\mathbf{X_{i-1}}$;
- θ_i : angle between axes $\mathbf{X_{i-1}}$, $\mathbf{X_i}$ corresponding to rotation about $\mathbf{Z_i}$;
- r_i : distance between $\mathbf{X_{i-1}}$, $\mathbf{X_i}$ along $\mathbf{Z_i}$.

The transformation matrix ${}^{\mathbf{i-1}}\mathbf{T_i}$ defining the frame R_i in the frame R_{i-1} is as follows [3, 4, 5]:

$${}^{\mathbf{i-1}}\mathbf{T_i} = \begin{bmatrix} \cos\theta_i & -\sin\theta_i & 0 & d_i \\ \cos\alpha_i \sin\theta_i & \cos\alpha_i \cos\theta_i & -\sin\alpha_i & -r_i \sin\alpha_i \\ \sin\alpha_i \sin\theta_i & \sin\alpha_i \cos\theta_i & \cos\alpha_i & r_i \cos\alpha_i \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Depending on the previous table and the matrix ${}^{\mathbf{i-1}}\mathbf{T_i}$ we define the transformation matrix as follows:

$$ {}^{\mathbf{0}}\mathbf{T_F} = {}^{\mathbf{0}}\mathbf{T_1}\,{}^{\mathbf{1}}\mathbf{T_2}\,{}^{\mathbf{2}}\mathbf{T_3}\,{}^{\mathbf{3}}\mathbf{T_F} = \begin{bmatrix} 0 & 0 & 1 & r_3 \\ 0 & 1 & 0 & r_2 \\ -1 & 0 & 0 & r_1 - d_F \\ 0 & 0 & 0 & 1 \end{bmatrix}. \quad (2) $$

Direct and inverse geometric model

The direct geometric model (DGM) represents the relations calculating the operational coordinates, giving the location of the Terminal link *F*, in terms of the joint coordinates [3, 4, 5]:

$$ \mathbf{X} = F(\mathbf{q}), $$

where the vector of motorized joint coordinates is: $\mathbf{q} = \begin{pmatrix} r_1 & r_2 & r_3 \end{pmatrix}^T$.

And the location of the Terminal link is: $\mathbf{X} = \begin{pmatrix} x_F & y_F & z_F \end{pmatrix}^T$.

We can find direct geometric model from equation (2), which is given by the following equations:

$$ x_F = r_3; \quad (3) $$

$$ y_F = r_2; \quad (4) $$

$$ z_F = r_1 - d_F. \quad (5) $$

And we can find the inverse geometric model from equation $\mathbf{q} = F^{-1}(\mathbf{X})$, which is given by the following equations:

$$ r_3 = x_F; \quad (6) $$

$$ r_2 = y_F; \quad (7) $$

$$ r_1 = z_F + d_F. \quad (8) $$

Workspace

Is set of points which can be occupied by point F (Terminal link). In our case the workspace is a cube shown in Figure 6.

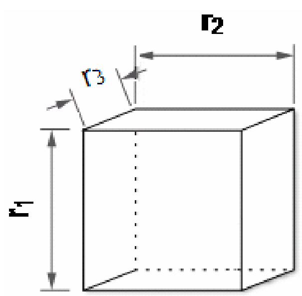

Figure 6. Workspace

Kinematic modeling

Direct kinematic model gives the velocities of Terminal link $\dot{\mathbf{X}} = \begin{pmatrix} \dot{x}_F & \dot{y}_F & \dot{z}_F \end{pmatrix}^T$ in terms of the joint velocities $\dot{\mathbf{q}} = \begin{pmatrix} \dot{d} & \dot{r}_2 & \dot{r}_3 \end{pmatrix}^T$, by using the Jacobi matrix $\mathbf{J}(q)$ according to the equation [3, 4, 5]:

$$ \dot{\mathbf{X}} = \mathbf{J}(q)\dot{\mathbf{q}} \Rightarrow \dot{\mathbf{q}} = \mathbf{J}^{-1}(q)\dot{\mathbf{X}}, $$

$$ J_{ij} = \frac{\partial F_i(q)}{\partial q_j}. $$

Direct and inverse kinematic model

By deriving the direct geometric model we find direct kinematic model:

$$ \dot{x}_F = \dot{r}_3; \quad (9) $$

$$ \dot{y}_F = \dot{r}_2; \quad (10) $$

$$ \dot{z}_F = \dot{r}_1. \quad (11) $$

And we can find inverse kinematic model by deriving inverse geometric model.

Jacobite matrix and anomalies

We conclude the Jacobi matrix from equations (9) and (10) and (11) that represent direct kinematic model:

$$ \mathbf{J} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}. \quad (12) $$

Singular configurations are defined as places in which inverse kinematic model cannot be determined. These configurations are the solutions for $\det(\mathbf{J}) = 0$.

We have: $\det(\mathbf{J}) = -1 \neq 0$. And therefore there are no Singular configurations for this robot.

Dynamic Modeling

The inverse dynamic model is the relation that gives the motors torques (and/or forces) in terms of the joint positions, velocities and accelerations. The inverse dynamic model is represented by equation (13) [3, 4, 5]:

$$ \mathbf{\Gamma} = f(\mathbf{q}, \dot{\mathbf{q}}, \ddot{\mathbf{q}}, \mathbf{F_e}), $$

where

$\mathbf{\Gamma}$: Vector of motors torques/forces, depending on whether the joint is revolute or prismatic;

$\mathbf{q}$: Vector of joint positions;

$\dot{\mathbf{q}}$: Vector of joint velocities;

$\ddot{\mathbf{q}}$: Vector of joint accelerations;

$\mathbf{F_e}$: Vector of external forces and moments that the environment exert on the robot.

The direct dynamic model expresses the joint accelerations in terms of joint positions, velocities and motors torques as in equation (14):

$$ \ddot{\mathbf{q}} = f(\mathbf{q}, \dot{\mathbf{q}}, \mathbf{\Gamma}, \mathbf{F_e}). \quad (14) $$

Dynamic model is used in the control processing and choosing the motors. To find direct and inverse dynamic model there are several methods:

Lagrange method that used in complex chain [3, 4, 5];

Newton – Euler method that used in serial chain [3, 4, 5].

In our case we have robot with serial chain so Newton_Euler method will be used.

Find the dynamic model by Newton – Euler method

To find the dynamic model, we depend on the recurrence way so to find the direct and inverse models we need two recurrences forward and backward [3, 4, 5].

In the **forward recurrence we** calculates the velocities and accelerations of the links starting from the base of the robot towards the terminal link, and then concludes the total forces and moments for each one of the links depending on equations (15) and (16).

In the **backward recurrence** we start from the terminal link towards the base. In each step we find motors torques by expressing for each link the resultant of external forces and moments.

• *Forward Recurrence*

Newton – Euler equation express the external force and torque on the link's center of mass C_i.

Newton – Euler relation to calculate the external force is [3, 4, 5]:

$$\mathbf{F_i} = M_i \dot{\mathbf{V}}_{\mathbf{G_i}}, \tag{15}$$

$\mathbf{F_i}$: The external forces acting on the link C_i;

M_i : Mass of Link C_i;

$\dot{\mathbf{V}}_{\mathbf{G_i}}$: Vector of linear acceleration of the center of mass of the link C_i.

Newton – Euler relation to calculate the external torque is [3, 4, 5]:

$$\mathbf{T_{G_i}} = \mathbf{I_{G_i}} \dot{\boldsymbol{\omega}}_\mathbf{i} + \boldsymbol{\omega}_\mathbf{i} \wedge \left(\mathbf{I_{G_i}} \boldsymbol{\omega}_\mathbf{i} \right), \tag{16}$$

$\mathbf{T_{G_i}}$: Total torque on the link C_i about his center of mass;

$\mathbf{I_{G_i}}$: Inertia matrix of the link C_i of his center of mass;

$\boldsymbol{\omega}_\mathbf{i}$: Angular velocity of the link C_i;

$\dot{\boldsymbol{\omega}}_\mathbf{i}$: Angular acceleration of the link C_i.

Note. Because all the joints in the Cartesian robot are prismatic joints, all rotational terms are equal to zero. Therefore we will find only the transitional terms, and we will calculate only the forces.

The force $\mathbf{F_i}$ is applied on O_i center of frame R_i so equation (15) can be written as follows:

$$\mathbf{F_i} = M_i \dot{\mathbf{V}}_\mathbf{i}, \tag{17}$$

$\dot{\mathbf{V}}_\mathbf{i}$: Acceleration of frame R_i Center.

Acceleration $\dot{\mathbf{V}}_\mathbf{i}$ is calculated from the equation (18) [3, 4, 5]:

$$\dot{\mathbf{V}}_\mathbf{i} = \dot{\mathbf{V}}_{\mathbf{i}-1} + \ddot{\mathbf{q}}_\mathbf{i} \mathbf{a_i}, \tag{18}$$

and $\mathbf{a_i}$ is a unit vector along axis $\mathbf{Z_i}$ expressed in frame R_0. Assuming that acceleration of link C_0 is

$$\dot{\mathbf{V}}_\mathbf{0} = \begin{bmatrix} 0 \\ 0 \\ -g \end{bmatrix},$$

we have:

$$\left\{ \begin{array}{l} \dot{\mathbf{V}}_1 = \begin{bmatrix} 0 \\ 0 \\ \ddot{r}_1 - g \end{bmatrix} \\ \dot{\mathbf{V}}_2 = \begin{bmatrix} 0 \\ \ddot{r}_2 \\ \ddot{r}_1 - g \end{bmatrix} \\ \dot{\mathbf{V}}_F = \dot{\mathbf{V}}_3 = \begin{bmatrix} \ddot{r}_3 \\ \ddot{r}_2 \\ \ddot{r}_1 - g \end{bmatrix} \end{array} \right\}. \tag{19}$$

After finding linear accelerations we can calculate $\mathbf{F_i}$ from the equation (17):

$$\left\{ \begin{array}{l} \mathbf{F}_1 = \begin{bmatrix} 0 \\ 0 \\ M_1(\ddot{r}_1 - g) \end{bmatrix} \\ \mathbf{F}_2 = \begin{bmatrix} 0 \\ M_2 \ddot{r}_2 \\ M_1(\ddot{r}_1 - g) \end{bmatrix} \\ \mathbf{F}_3 = \begin{bmatrix} M_3 \ddot{r}_3 \\ M_2 \ddot{r}_2 \\ M_1(\ddot{r}_1 - g) \end{bmatrix} \end{array} \right\}. \tag{20}$$

• *Backward recurrence*

Figure 7 shows the forces acting on the link C_i.

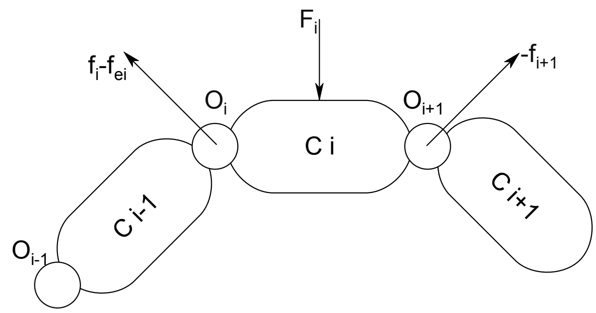

Figure 7. The forces acting on the link C_i : $\mathbf{f_i}$ – the force applied by the link C_{i-1} on the link C_i; $\mathbf{f_{ei}}$ – the force applied by the environment on the link C_i

By calculating $\mathbf{f_i}$ and adding the effect of the friction, the force required from the motors can be calculated assuming the motor is fixed on the link C_{i-1} and moves the link C_i. The expression of $\mathbf{f_i}$ (equation 21) can be concluded from Figure7:

$$\mathbf{f_i} = \mathbf{F_i} + \mathbf{f_{i+1}} + \mathbf{f_{ei}}. \quad (21)$$

We start calculating from terminal link which is fixed to link C_3 toward to link C_1, taking into account the following considerations:

- $\mathbf{f_{ei}}$ equal to zero;
- $\mathbf{f_{n+1}} = \mathbf{f_L} = M_L \begin{bmatrix} 0 \\ 0 \\ -g \end{bmatrix}$;

M_L : Load mass;

$\mathbf{f_L}$: is the force which is applied to the terminal link of the robot:

$$\begin{Bmatrix} \mathbf{f_3} = \mathbf{F_3} + \mathbf{f_L} = \begin{bmatrix} (M_3)\ddot{r}_3 \\ (M_3)\ddot{r}_2 \\ (M_3)(\ddot{r}_1 - g) - M_L g \end{bmatrix} \\ \mathbf{f_2} = \mathbf{F_2} + \mathbf{f_3} = \begin{bmatrix} (M_3)\ddot{r}_3 \\ (M_2 + M_3)\ddot{r}_2 \\ (M_2 + M_3)(\ddot{r}_1 - g) - M_L g \end{bmatrix} \\ \mathbf{f_1} = \mathbf{F_1} + \mathbf{f_2} = \begin{bmatrix} (M_3)\ddot{r}_3 \\ (M_2 + M_3)\ddot{r}_2 \\ (M_1 + M_2 + M_3)(\ddot{r}_1 - g) - M_L g \end{bmatrix} \end{Bmatrix} \quad (22)$$

Calculating motors torques

After calculating $\mathbf{f_i}$ and adding the effect of friction, the force Γ_i required from the motors can be calculated using equation (23) [3, 4, 5]:

$$\Gamma_i = \mathbf{f}_i \mathbf{a_i} + F_{f_i}, \quad (23)$$

where

F_{f_i} : The friction in the joint. Equation (24) expresses Γ_i :

$$\begin{Bmatrix} \Gamma_1 = \mathbf{f_1}\,\mathbf{a_1} + F_{f_1} \\ = (M_1 + M_2 + M_3)(\ddot{r}_1 - g) - M_L g + F_{f_1} \\ \Gamma_2 = \mathbf{f_2}\,\mathbf{a_2} + F_{f_2} \\ = (M_2 + M_3)\ddot{r}_2 - (M_2 + M_3 + M_L) g\mu_2 \\ \Gamma_3 = \mathbf{f_2}\mathbf{a_3} + F_{f_3} \\ = (M_3)\ddot{r}_3 - (M_3 + M_L) g\mu_3 \end{Bmatrix}. \quad (24)$$

Because all joints acceleration values can be neglected, equation (24) becomes as follows:

$$\begin{Bmatrix} \Gamma_1 = f_1 + F_{f_1} \\ = -(M_1 + M_2 + M_3 + M_L)(g) - F_{f_1} \\ \Gamma_2 = f_2 + F_{f_2} \\ = -(M_2 + M_3 + M_L)(g\mu_2) \\ \Gamma_3 = f_3 + F_{f_3} \\ = -(M_3 + M_L)(g\mu_3) \end{Bmatrix}. \quad (25)$$

Movement is transmitted from the motor to the link through a pinion-rack, so for calculating the required torque we multiply the force calculated from equation (27) by radius of the pinion fixed with motor as follows:

$$\begin{Bmatrix} T_{m1} = \Gamma_1 R_1 \\ T_{m2} = \Gamma_2 R_2 \\ T_{m3} = \Gamma_3 R_3 \end{Bmatrix}. \quad (26)$$

T_{mi} : i^{th} Motor torque;

R_i : i^{th} the radius of the dentate fixed with motor.

Note: Due to the difficulty in determining friction F_{f_1}, we try to overcome this problem by multiplying the value of the calculated torque by a safety factor equals to $\eta = 1.5$. Finally, find the equations (29) that express the values of motors torques:

$$\begin{Bmatrix} T_{m1} = (M_1 + M_2 + M_3 + M_L)(g)(R_1)\eta \\ T_{m2} = (M_2 + M_3 + M_L)(g\mu_2)(R_2) \\ T_{m3} = (M_3 + M_L)(g\mu_3)(R_3) \end{Bmatrix}. \quad (27)$$

Results

We used Mathematica 8.0 to draw the function $T_{mi} = F(M_L)$ which express the torque in terms of the load mass.

Note the torque lines don't intersect with the frame center (0,0) because the motor torque doesn't equal to zero in the case of no load ($M_L = 0$), and that happened due to the need to overcome link mass.

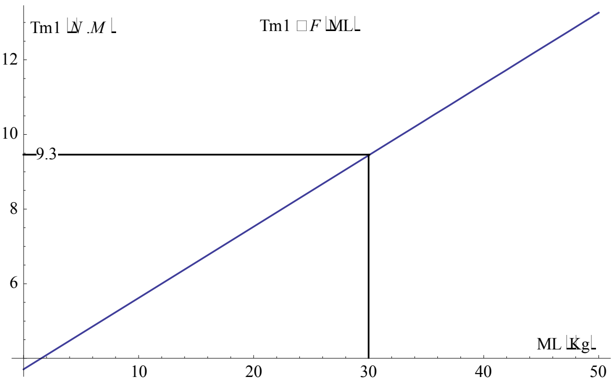

Figure 8

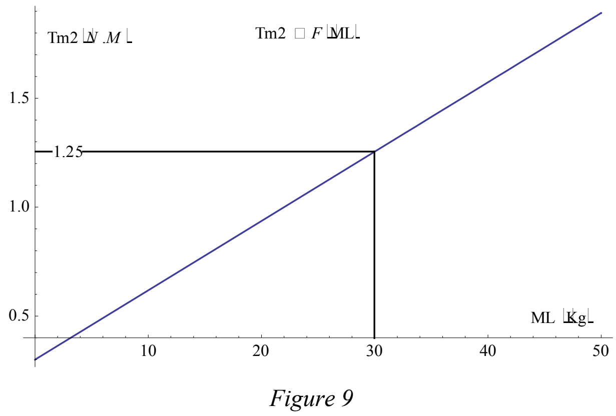

Figure 9

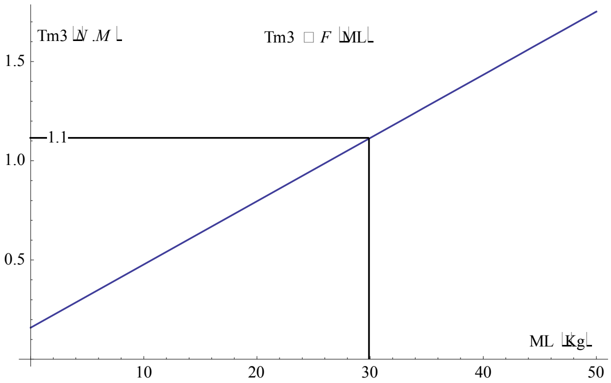

Figure 10

By manually calculating the torques from equation (27) for load mass equal to 30 Kg, we find:

$$\begin{Bmatrix} T_{m1} = 9.3\,N.m \\ T_{m2} = 1.25\,N.m \\ T_{m3} = 1.1\,N.m \end{Bmatrix}. \qquad (28)$$

These results are identical with the values in Figures 8, 9 and 10.

And the rest of the specifications are shown in Table 2.

We note from the previous table that torque can be calculated by knowing the force to be overcome and the radius of the pinion.

Table 2. **Specifications the motors**

Power $P = T.\omega$ (W)	Angular speed $\omega = (30.V)/(\pi.R)$ (RPM)	Linear speed $V = L/S$ $(m.s^{-1})$	Time $S(s)$	Length $L(m)$	Torque $T = F.R$ $(N.m)$	Radius $R(m)$	Force $F(N)$
14.3	15	0.02	30	0.6	9.3	0.013	$478 + F_{f_i}$
1.93	15	0.02	30	0.6	1.25	0.013	96.5
1.7	15	0.02	10	0.2	1.1	0.013	85.7

After calculating the angular speed and torque we can determine the motor power.

References

1. International Assessment of Research and Development in Robotics / George Bekey, Robert Ambrose, Vijay Kumar, Art Sanderson, Brian Wilcox, Yuan Zheng. – WTEC, 2006.

2. *Norton Robert L.* Design of Machinery. – McGraw-Hill, 1999.

3. *Dombre Etienne*, *Khalil Wisam.* Robot Manipulators. – ISTE, 2007.

4. *Siciliano Bruno*, *Khatib Oussama.* Springer Handbook of Robotics. – Springer, 2008.

5. *Craig John J.* Introduction to Robotics: Mechanics and Control. – 2005.

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Геометрическое, кинематическое и динамическое моделирование декартова робота

Целью моделирования любого робота является получение его функции, наиболее точно согласующейся с его назначением. В процессе моделирования робота необходимо найти уравнения, определяющие его геометрические, кинематические и динамические переменные. В этой статье описан робот, работающий в декартовой системе координат (с тремя степенями свободы), и получены геометрическая, кинематическая и динамическая модели. Сначала используется геометрическая модель для определения рабочего пространства робота. Затем с помощью кинематической модели находятся линейные скорости движения звеньев робота, а также изучены особенности его конфигурации. Наконец, разрабатывается динамическая модель для расчета сил и крутящих моментов, необходимых для выбора двигателей, приводящих в движение звенья робота.

Ключевые слова: геометрическое моделирование, кинематическое моделирование, динамическое моделирование, декартов робот.