ИНФОРМАТИКА, ВЫЧИСЛИТЕЛЬНАЯ ТЕХНИКА И УПРАВЛЕНИЕ

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DESIGING A REAL MATHEMATICAL MODEL OF A HEXACOPTER IN THE INERTIAL FRAME

1. Introduction

Nowadays, the mini-drones invade several application domains [1][2][3]. The control of an aerial robot such as a hexacopter requires studying its dynamics in order to account for gravity effects and aerodynamic forces [1][2]. These aerial vehicles have high maneuverability and stationary flight [4][5]. In this work equations of motion were derived of the whole system using the Newton-Euler formulation for translational and rotational dynamics of a rigid body [1]. This paper is focusing on studying the nature of characteristics such as nonlinearity and coupling of variables, then adding disturbances that are presented as an environment for outdoor simulation [3], which is omitted in most of the literature. The structure of the paper is as follows: describing both the kinematics and dynamics then explaining the equations of motion.

2. Reference Systems for the UAV Hexacopter

In order to describe the hexacopter motion only two reference systems are necessary: earth inertial frame (E-frame) and body-fixed frame (B-frame). An Inertial frame is a system that uses the North, East, and Down (NED) coordinates and the origin of this reference system is fixed in one point located on the earth surface as shown in Figure 1, and the (X, Y, Z) axes are directed to the North, East, and Down, respectively. The mobile frame (X_B, Y_B, Z_B) is the body fixed frame that is centered in the hexacopter center of gravity and oriented as shown in Figure 1. The angular position of the body frame with respect to the inertial one is usually defined by means of the Euler angles: roll φ, pitch θ, and yaw ψ. As the vector: $\sigma = [\varphi\ \theta\ \psi]^{\mathrm{T}}$, φ and $\theta \in]-\pi/2, -\pi/2[$; $\psi \in]-\pi, \pi[$. The inertial frame position of the vehicle is given by vector $\xi = [x\ y\ z]^{\mathrm{T}}$ [1] [2] [5]. The transformation from the body frame to the inertial frame is realized by using the well-known rotation matrix C_b^n [7] [8]:

$$C_b^n = \begin{bmatrix} c\theta c\psi & s\varphi s\theta c\psi - c\varphi s\psi & s\varphi s\psi + c\varphi s\theta c\psi \\ c\theta s\psi & c\varphi c\psi + s\varphi s\theta s\psi & c\varphi s\theta s\psi - s\varphi c\psi \\ -s\theta & s\varphi c\theta & c\varphi c\theta \end{bmatrix},$$

which is orthogonal, and $c\theta$ equivalent to cosθ also $s\theta$ means sinθ, while the transformation matrix for angular velocities from the body frame to the inertial one is S [3][5]. Where $\dot{\sigma} = S\Omega$, $\dot{\xi} = C_b^n V$, the angular velocity Ω is defined by the vector $\Omega = [p\ q\ r]^{\mathrm{T}}$, and the linear velocity is defined by the vector $V = [u\ v\ w]^{\mathrm{T}}$ in the body frame.

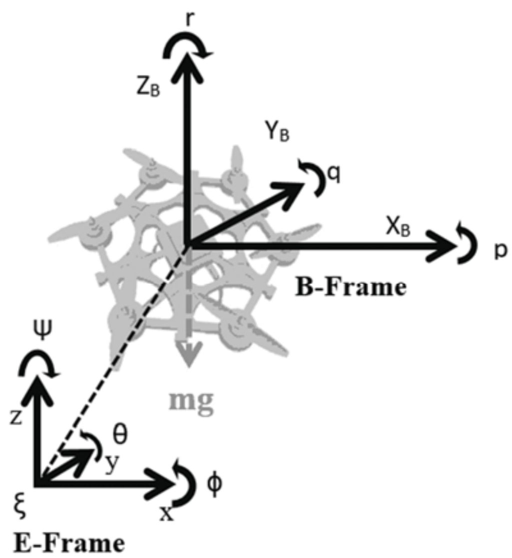

Figure 1. UAV, Hexacopter Structure and Frames

3. Aerodynamic Forces and Moments in Axial Flight

In order to describe the dynamics of the hexacopter, that is assumed to be a rigid body and has a symmetrical structure, Newton-Euler equations [1][3], that govern linear and angular motion are used as shown in equation 2, where m is the mass of hexacopter, $\sum F$ is the total force and $\sum M$ is the total moment acting along the axis:

$$\begin{bmatrix} mI_{3\times3} & 0_{3\times3} \\ 0_{3\times3} & J \end{bmatrix}\begin{bmatrix} \dot{V} \\ \dot{\Omega} \end{bmatrix} + \begin{bmatrix} \Omega\times(mV) \\ \Omega\times(J\Omega) \end{bmatrix} = \begin{bmatrix} \sum F \\ \sum M \end{bmatrix}.$$

3.1. Force Analysis

a. Thrust Force: The main force affecting the aircraft movement is the thrust force resulting from the propellers that lead to raising the aircraft in the air. The model has the total thrust force vector is T which is the sum of the propellers thrust force vectors $T=\sum_{i=1}^{6}T_i$ in the body frame. The thrust and torque as in [5] are: $|T_i|=\rho C_T AR^2\omega_i^2$, and $|Q_i|=\rho C_Q AR^3\omega_i^2$. Where blade rotation is with angular velocity ω, the blade radius is R, C_T and C_Q are respectively thrust and torque coefficients, ρ is the air density and A the disc area. The thrust and torque coefficients can be written as:

$$C_T=\frac{1}{4}\mu C_{L\alpha}\left[\frac{2\theta_b}{3}-\left(\gamma_c+\gamma_i\right)\right],$$

$$C_Q=(1/2)\mu\times$$
$$\times\left[C_{L\alpha}\left(\gamma_c+\gamma_i\right)\left\{(\theta_b/3)-\left((\gamma_c+\gamma_i)/2\right)\right\}+\left(C_D/4\right)\right],$$

where μ is the rotor solidity, $C_{L\alpha}$ is the lift slope coefficient, C_D is the drag coefficient γ_c, and γ_i are the inflow factors [5]. Finally, the total force of thrust generated by the six propellers in the earth frame is defined as:

$$F_{TI}=C_b^n\begin{bmatrix}0\\0\\\sum_{i=1}^{6}|T_i|\end{bmatrix}_B=\begin{bmatrix}(c\varphi c\psi s\theta+s\varphi s\psi)\sum_{i=1}^{6}|T_i|\\(c\varphi s\theta s\psi-s\varphi c\psi)\sum_{i=1}^{6}|T_i|\\(c\varphi c\theta)\sum_{i=1}^{6}|T_i|\end{bmatrix}_E.$$

b. Drag Force: It is the opposing force to the travelling of the hexacopter in air, which is resulting from the aerodynamic friction, air density, velocity, and can be expressed by the following equation at the earth's frame: $F_{AI}=K_{TI}\xi$, where K_{TI} is a diagonal matrix related to the aerodynamic friction constant k_t [4][5].

c. Gravitational Force: The gravity force is directed toward the center of the earth and the equation of the gravity force in the earth frame according to [1][2] is: $F_{GI}=m[0\ 0\ g]_E^{\mathrm{T}}$.

d. Disturbance Force: Other forces like the Coriolis force from the earth, the wind, and Euler forces are considered as a disturbance, summarized as F_{DI} in the earth frame: $F_{DI}=\left[F_{dIx}\ F_{dIy}\ F_{dIz}\right]_E^{\mathrm{T}}$. Therefore, the equations of motion that govern the translational motion with respect to the earth frame are:

$$\sum F=F_{TI}-F_{AI}-F_{GI}+F_{DI}=mV+\Omega\times(mV);$$

$$\begin{bmatrix}(c\varphi c\psi s\theta+s\varphi s\psi)\sum_{i=1}^{6}|T_i|\\(c\varphi s\theta s\psi-s\varphi c\psi)\sum_{i=1}^{6}|T_i|\\(c\varphi c\theta)\sum_{i=1}^{6}|T_i|\end{bmatrix}_E-\begin{bmatrix}k_t\dot{x}\\k_t\dot{y}\\k_t\dot{z}\end{bmatrix}_E-$$
$$-m\begin{bmatrix}0\\0\\g\end{bmatrix}_E+\begin{bmatrix}F_{dIx}\\F_{dIy}\\F_{dIz}\end{bmatrix}_E=m\begin{bmatrix}\ddot{x}\\\ddot{y}\\\ddot{z}\end{bmatrix}_E,$$

where $\Omega\times(mV)$ has a small effect and approximately equal to zero. Therefore, the equations, after some simplifications, are as follows:

$$\begin{pmatrix}\ddot{x}=U_x-\frac{k_t}{m}\dot{x}+\frac{F_{dIx}}{m}\\\ddot{y}=U_y-\frac{k_t}{m}\dot{y}+\frac{F_{dIy}}{m}\\\ddot{z}=U_z-\frac{k_t}{m}\dot{z}+\frac{F_{dIz}}{m}\end{pmatrix},$$

where

$$U_x=(c\varphi c\psi s\theta+s\varphi s\psi)u_T/m=a_x(\varphi,\theta,\psi)\sum_{i=1}^{6}\omega_i^2;$$

$$U_y=(c\varphi s\theta s\psi-s\varphi c\psi)u_T/m=a_y(\varphi,\theta,\psi)\sum_{i=1}^{6}\omega_i^2;$$

$$U_z=(c\varphi c\theta)u_T/m=a_z(\varphi,\theta)\sum_{i=1}^{6}\omega_i^2;$$

$$u_T=\sum_{i=1}^{6}|T_i|=\rho C_T AR^2\sum_{i=1}^{6}\omega_i^2;\quad a=\frac{k_t}{m}\to 0.$$

Therefore, the final equations with respect to Earth frame are:

$$\begin{bmatrix}\ddot{x}\\\ddot{y}\\\ddot{z}\end{bmatrix}=\begin{bmatrix}-a\dot{x}+U_x\\-a\dot{y}+U_y\\-a\dot{z}+U_z-g\end{bmatrix}+\begin{bmatrix}\frac{F_{dIx}}{m}\\\frac{F_{dIy}}{m}\\\frac{F_{dIz}}{m}\end{bmatrix}=\begin{bmatrix}U_x\\U_y\\U_z-g\end{bmatrix}+\begin{bmatrix}\frac{F_{dIx}}{m}\\\frac{F_{dIy}}{m}\\\frac{F_{dIz}}{m}\end{bmatrix}.$$

3.2. Moments Analysis

The aircraft is affected by several types of moments: the thrust moment resulting from the motors, the motors inertia moment, the aerodynamic moment, and the disturbances moment. Supposing, the inertia matrix of the aircraft is J, the structure of the aircraft is symmetric, so according to [5] the inertia matrix is of the following form: $J = \left[J_{xx}\ 0\ 0;\ 0\ J_{yy}\ 0;\ 0\ 0\ J_{zz} \right]^{\mathrm{T}}$; $J \in R_{3\times3}$ [5]. Therefore, the moments acting on the center of the aircraft can be analyzed as follows:

a. Propeller Moments: The M_{thrust} is part of the external moments, and is described by the propeller thrust $\sum_{i=1}^{6} T_i$ generated by the propellers, and the distance l from CG to the center of the propeller. The attitude of the vehicle in the air, i. e., Euler angles $\sigma = [\varphi\ \theta\ \psi]^{\mathrm{T}}$ change, by controlling the angular velocity of motors, i. e., there is a different thrust moment over time, $M_T = \left[M_p\ M_q\ M_r \right]^{\mathrm{T}}$, where M_p, M_q, M_r are the moments about the axes X_B, Y_B, Z_B in the body frame [1][3] [4], therefore M_T is described by the follow expression:

$$M_T = \begin{bmatrix} \frac{\sqrt{3}}{2} l\left(|T_3| - |T_4| - |T_5| + |T_6|\right) \\ \frac{1}{2} l\left(|T_3| - |T_4| + |T_5| - |T_6| + 2|T_1| - 2|T_2|\right) \\ \rho C_Q A R^3 \left(\omega_1^2 + \omega_4^2 + \omega_6^2 - \omega_2^2 - \omega_3^2 - \omega_5^2\right) \end{bmatrix}.$$

b. The aerodynamic moment: It is the moment resulting from the aerodynamic friction in air and is proportional to the torque around the axes, and it is expressed by the following equation: $M_{AI} = K_{RI}\Omega^2 = K_{RI}\left[\dot{\varphi}^2\ \dot{\theta}^2\ \dot{\psi}^2 \right]^{\mathrm{T}}$, where K_{RI} is a diagonal matrix related to the rotational aerodynamic friction constant by the parameter K_r [4][5].

c. Disturbance moment: It is the total of the disturbances affecting the torque around the aircraft axes resulting from disturbances in the motors movement, the wind, and the load in the aircraft, expressed as $M_{DI} = \left[M_{dI\varphi}\ M_{dI\theta}\ M_{dI\psi} \right]^{\mathrm{T}}$.

d. Propeller Gyroscopic effect: The rotation of the propellers produces a gyroscopic effect: $M_{gyro} = \left[-J_r\dot{\theta}\omega_r\ J_r\dot{\varphi}\omega_r\ 0 \right]^{\mathrm{T}}$ [3], where J_r is the rotational inertia of the propeller [NmS^2] and the ω_r is the overall propeller speed [rad/s], as $\omega_r = -\omega_1 + \omega_2 - \omega_3 + \omega_4 - \omega_5 + \omega_6$.

e. Yaw counter moment: Differences in rotational acceleration of the propellers produces a yaw inertial counter moment as follow: $M_{counter} = \left[0\ 0\ J_r\dot{\omega}_r\right]^{\mathrm{T}}$ [3]. Therefore, the equations of motion that govern the rotational motion with respect to the body frame are:

$$\sum M = M_T - M_{AI} + M_{gyro} + M_{counter} + M_{DI} = = J\dot{\Omega} + \Omega \times (m\Omega);$$

$$\begin{bmatrix} M_p \\ M_q \\ M_r \end{bmatrix} - \begin{bmatrix} k_r\dot{\varphi}^2 \\ k_r\dot{\theta}^2 \\ k_r\dot{\psi}^2 \end{bmatrix} + \begin{bmatrix} -J_r\dot{\theta}\omega_r \\ J_r\dot{\varphi}\omega_r \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ J_r\dot{\omega}_r \end{bmatrix} + \begin{bmatrix} M_{dI\varphi} \\ M_{dI\theta} \\ M_{dI\psi} \end{bmatrix} = = J\begin{bmatrix} \dot{p} \\ \dot{q} \\ \dot{r} \end{bmatrix} + \begin{bmatrix} p \\ q \\ r \end{bmatrix} \times J \begin{bmatrix} p \\ q \\ r \end{bmatrix};$$

$$\begin{pmatrix} \dot{p} = \frac{M_p}{J_x} + qr\frac{(J_y - J_z)}{J_x} - \frac{k_r}{J_x}p - \frac{J_r}{J_x}\dot{\theta}\omega_r + \frac{M_{dI\varphi}}{J_x} \\ \dot{q} = \frac{M_q}{J_y} + pr\frac{(J_z - J_x)}{J_y} - \frac{k_r}{J_y}q - \frac{J_r}{J_y}\dot{\varphi}\omega_r + \frac{M_{dI\varphi}}{J_y} \\ \dot{r} = \frac{M_r}{J_z} + pq\frac{(J_x - J_y)}{J_z} - \frac{k_r}{J_z}r - \frac{J_r}{J_z}\dot{\omega}_r + \frac{M_{dI\psi}}{J_z} \end{pmatrix};$$

$$\dot{\sigma} = \begin{bmatrix} \dot{\varphi} \\ \dot{\theta} \\ \dot{\psi} \end{bmatrix}_E = S\begin{bmatrix} p \\ q \\ r \end{bmatrix}_B .$$

When $S \to I$ the hexacopter tends to the stable point, therefore the equations of angular rate will be related to the earth frame, in addition to considering some assumptions as: $\frac{J_r}{J_x} = \frac{J_r}{J_y} = \frac{J_r}{J_z} \to 0$ is a small effect around the zero, and $J_x = J_y$, so the equations will be as shown in the following:

$$\begin{pmatrix} \ddot{\varphi} = U_p + b_1\dot{\theta}\dot{\psi} + c_1\dot{\varphi}^2 + \frac{M_{dI\varphi}}{J_x} \\ \ddot{\theta} = U_q + b_2\dot{\varphi}\dot{\psi} + c_2\dot{\theta}^2 + \frac{M_{dI\theta}}{J_y} \\ \ddot{\psi} = U_r + c_3\dot{\psi}^2 + \frac{M_{dI\psi}}{J_z} \end{pmatrix};$$

$$\begin{cases} U_p = \frac{M_p}{J_x} = \frac{\sqrt{3}\rho l c_T A R^2}{2J_x}\left(\omega_3^2 + \omega_6^2 - \omega_4^2 - \omega_5^2\right), \\ U_p = a_\varphi\left(\omega_3^2 + \omega_6^2 - \omega_4^2 - \omega_5^2\right), \\ U_q = \frac{M_q}{J_y} = \frac{\rho l c_T A R^2}{2J_y}\left(\omega_3^2 + \omega_5^2 + 2\omega_1^2 - \omega_4^2 - \omega_6^2 - 2\omega_2^2\right), \\ U_q = a_\theta\left(\omega_3^2 + \omega_5^2 + 2\omega_1^2 - \omega_4^2 - \omega_6^2 - 2\omega_2^2\right), \\ U_r = \frac{M_\gamma}{J_z} = \frac{\rho l c_Q A R^3}{2J_z}\left(\omega_1^2 + \omega_4^2 + \omega_6^2 - \omega_2^2 - \omega_3^2 - \omega_5^2\right), \\ U_r = a_\psi\left(\omega_1^2 + \omega_4^2 + \omega_6^2 - \omega_2^2 - \omega_3^2 - \omega_5^2\right), \\ b_1 = \frac{(J_y - J_z)}{J_x},\ b_2 = \frac{(J_z - J_x)}{J_y}, \\ c_1 = \frac{k_r}{J_x},\ c_2 = \frac{k_r}{J_y},\ c_3 = \frac{k_r}{J_z}. \end{cases} \quad (17)$$

4. State Space Model

The dynamic model presented in translational and rotational equation set can be rewritten in the state-space form: $\dot{X} = f(X) + g(X,U) + \delta$, where δ is the disturbances, and the $X \in \Re^{12}$ is the vector of state variables given as follows:

$$X = \begin{bmatrix} x & \dot{x} & y & \dot{y} & z & \dot{z} & \varphi & \dot{\varphi} & \theta & \dot{\theta} & \psi & \dot{\psi} \end{bmatrix}^{\mathrm{T}}.$$

Then we can derive the final equations of the system, which governs the transitional and rotational of hexacopter, with respect to the earth frame in state space as follows:

$$\begin{pmatrix} \dot{x}_2 \\ \dot{x}_4 \\ \dot{x}_6 \end{pmatrix} = \begin{pmatrix} -a & & \\ & -a & \\ & & -a \end{pmatrix} \begin{pmatrix} x_2 \\ x_4 \\ x_6 \end{pmatrix} + \begin{pmatrix} U_x \\ U_y \\ U_z \end{pmatrix} + \begin{pmatrix} F_{dIx}/m \\ F_{dIy}/m \\ \dfrac{F_{dIz}}{m} - g \end{pmatrix} =$$

$$= \begin{pmatrix} U_x \\ U_y \\ U_z \end{pmatrix} + \begin{pmatrix} F_{dIx}/m \\ F_{dIy}/m \\ \dfrac{F_{dIz}}{m} - g \end{pmatrix};$$

$$\begin{pmatrix} \dot{x}_8 \\ \dot{x}_{10} \\ \dot{x}_{12} \end{pmatrix} = \begin{pmatrix} b_1 x_{10} x_{12} + c_1 x_8^2 \\ b_2 x_8 x_{12} + c_2 x_{10}^2 \\ b_3 x_8 x_{10} + c_3 x_{12}^2 \end{pmatrix} +$$

$$+ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} U_p \\ U_q \\ U_r \end{pmatrix} + \begin{pmatrix} \dfrac{M_{dI\varphi}}{J_x} \\ \dfrac{M_{dI\theta}}{J_y} \\ \dfrac{M_{dI\psi}}{J_z} \end{pmatrix}.$$

The control input to the system is the vector $U = \begin{bmatrix} U_x & U_y & U_z & U_p & U_q & U_r \end{bmatrix}^{\mathrm{T}}$, where the following equation gives the relation between the angular speed of the propellers and the control inputs.

$$\begin{bmatrix} U_x \\ U_y \\ U_z \\ U_p \\ U_q \\ U_r \end{bmatrix} = \begin{bmatrix} +a_x & +a_x & +a_x & +a_x & +a_x & +a_x \\ +a_y & +a_y & +a_y & +a_y & +a_y & +a_y \\ +a_z & +a_z & +a_z & +a_z & +a_z & +a_z \\ 0 & 0 & +a_\varphi & -a_\varphi & -a_\varphi & +a_\varphi \\ 2a_\theta & -2a_\theta & +a_\theta & -a_\theta & +a_\theta & -a_\theta \\ +a_\psi & -a_\psi & -a_\psi & +a_\psi & -a_\psi & +a_\psi \end{bmatrix} \begin{bmatrix} \omega_1^2 \\ \omega_2^2 \\ \omega_3^2 \\ \omega_4^2 \\ \omega_5^2 \\ \omega_6^2 \end{bmatrix}.$$

The final concluded equations were represented the nonlinearity, coupled and time-variance in variables that are not considered in other papers like [3][4], the others are used simplification techniques as well as linearization tools. In addition, the suggested model was taking into consideration the disturbances in forces and moments.

5. Conclusion

The suggested real and complex dynamic mathematical model, derived as shown in the equations, is suitable for systems that are characterized by nonlinearity, time variance, and coupling among their variables, and for outdoor applications. Any change in the input variables leads to changes in most of the output variables, where additionally disturbances were added the thing that was not considered in other papers. With respect to the control input vector U, it is clear that the rotational subsystem is fully-actuated, it is only dependent on rotational state variables x_7 to x_{12}, while the translational subsystem is under-actuated as it is dependent on both the translational state variables x_1 to x_6 and the rotational ones x_7 to x_{12}.

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